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Solving A simple Harmonic Oscillator Equation by Adomian Decomposition Method

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Abstract: **In this paper, we will display Adomian decomposition method (ADM) for solving a simple harmonic oscillator equation· It is shown that the Adomian decomposition method (ADM) efficiency, simple, easy to use in solving physical equation. The proposed method can be applied to linear problem·**

Some examples were presented to show the ability of the method for linear ordinary differential physical equations·

Keywords: **Adomian decomposition method, harmonic oscillator schordinger equation, physical equation.**

1. INTRODUCTION

The harmonic oscillator is one of the most important model system in equation mechaincs. An harmonic oscillator is a particale subject to a restoring force that is proportional. to displacement of the particale [4]· In classical physics this means,

$$F = ma = m\frac{dy^2}{dt^2} = -K\,x \qquad (1)\cdot$$

The costant k is known the force constant· The study of quantum harmonic motion be beginning with specification of the schrodinger equation· We can write the schrodinger equation for a simple harmonic oscillator [5,6,7]·

$$(\frac{-h^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2)\varphi(x) = E\varphi(x) \qquad (2)\cdot$$

$$\text{Where } x = 0, \text{the equation becomes as, } \frac{d^2\varphi(x)}{dx^2} + \frac{2mE}{h^2}\varphi(x) = 0 \qquad (3)\cdot$$

Hence, the Schrodinger equation becomes as

$$\frac{d^2\varphi(x)}{dx^2} + \beta^2\varphi(x) = 0 \qquad (4)\cdot$$

$$\text{Where } \beta = \sqrt{\frac{2mE}{h}}$$

In this paper we will using Adomian decomposition method to solving a simple harmonic oscillator and schordinger for some simple harmonic oscillator equations.

2. ANALYSIS OF THE METHOD

Under the transformation $\omega = \sqrt{\frac{k}{m}}$ the equation (1) is transformer to

$$\frac{d^2y}{dt^2} = -\omega^2 y \qquad (5)\cdot$$

Where k force constant and m is mass the equation (4) becomes

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad (6)\cdot$$

The equation (6) is called a simple harmonic oscillator.

We propose the new differential operator [1,2,9,10], as below

$$L(\cdot) = e^{i\omega t}\frac{d}{dt}e^{-2i\omega t}\frac{d}{dt}e^{i\omega t}(\cdot) \quad (7),$$

so, the problem (6) can be written as

$$L(\cdot) = 0 \qquad (8)\cdot$$

The inverse operator L^{-1} is therefor considered a tow-fold integeral operator, as below

$$L^{-1}(\cdot) = e^{-i\omega t}\int_0^t e^{2i\omega t}\int_0^t e^{-i\omega t}(y'' + \omega^2 y) \quad (9)\cdot$$

Applying L^{-1} of (9) to the tow terms $y'' + \omega^2 y$ of Eq $\cdot$ (6) we find

$$L^{-1}(y'' + \omega^2 y)$$

$$= e^{-i\omega t}\int_0^t e^{2i\omega t}\int_0^t e^{-i\omega t}(y'' + \omega^2 y)dtdt$$

$$= e^{-i\omega t}\int_0^t e^{2i\omega t}(e^{-i\omega t}y' + i\omega e^{-i\omega t} - y'(0) - i\omega y(0)$$

$$= y - e^{-i\omega t}y(0) - \frac{y'(0)}{2i\omega}e^{i\omega t} + \frac{y'(0)}{2i\omega}e^{-i\omega t} - \frac{y(0)}{2}e^{i\omega t} + \frac{y(0)}{2}e^{-i\omega t},$$

Operating with L^{-1} on (6), it follows

$$y(t) = e^{-i\omega t}y(0) + \frac{y'(0)}{2i\omega}e^{i\omega t} - \frac{y'(0)}{2i\omega}e^{-i\omega t} + \frac{y(0)}{2}e^{i\omega t} - \frac{y(0)}{2}e^{-i\omega t} \qquad (10)\cdot$$

The Adomian decomposition method introduse the solution y (t),

so, the exact solution is easily obtioned by this method·

3. APPLICATION OF A SIMPLE HARMONIC OSCILLATOR (SHO)

3-1· Simple pendulum

Case1: if θ smaller then sin θ=θ.

Consider the equation of motion of a pendulum with length (L) and angle θ

from the vertical to the pendulum· We can be shown that θ, as a function of time satisfied the nonlinear differential equations [8]·

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0 \qquad (11)\cdot$$

Where g is the acceration to gravity. For small values of θ ·

We can use linear approximation sinθ = θ and then the equation becomes linear.

The Eq· (11) becomes

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\theta=0 \qquad (12)\cdot \text{We put } \gamma=\sqrt{\frac{g}{L}}, \text{this implies}$$

$$\theta''+\gamma^2\theta=0 \qquad (13)\cdot$$

With intial value $\theta(0)=1, \theta'(0)=0$,

we put $\omega=\gamma$,substitution of $\gamma=\omega$ in Eq $\cdot$ (6) yield the operator

$$L(\cdot)=e^{i\gamma t}\frac{d}{dt}e^{-2i\gamma t}\frac{d}{dt}e^{i\gamma t}(\cdot) \qquad (14),$$

so

$$L^{-1}(\cdot)=e^{-i\gamma t}\int_0^t e^{2i\gamma t}\int_0^t e^{-i\gamma t}(\cdot)dtdt \qquad (15)\cdot$$

In an operator from, Eq· (13) becomes

$$L\theta=0 \quad (16)\cdot$$

Applying L^{-1} on both sides of (16) we find

$$L^{-1}L\theta=0,$$

and implies,

$$\theta(t)=e^{-i\gamma t}\theta(0)+\frac{\theta'(0)}{2i\gamma}e^{i\gamma t}-\frac{\theta'(0)}{2i\gamma}e^{-i\gamma t}+\frac{\theta(0)}{2}e^{i\gamma t}-\frac{\theta(0)}{2}e^{-i\gamma t},$$

$$=\frac{1}{2}e^{i\gamma t}+\frac{1}{2}e^{-i\gamma t}=\cos\gamma t,$$

so, the exact solution is easily obtained by this method·

Example (3-2)

Case2: if θ is larger the sin (π+θ) = -θ,

the Eq· (11) becomes,

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin(\pi+\theta)=0 \qquad (17)\cdot$$

This equation becomes,

$$\theta''-\alpha^2\theta=0 \qquad (18)\cdot$$

Where $\alpha=\sqrt{\frac{g}{L}}$ and intial value $\theta(0)=1, \theta'(0)=\alpha\cdot$

We propose the new differential operator,

as, below,

$$L(\cdot)=e^{\alpha t}\frac{d}{dt}e^{-2\alpha t}\frac{d}{dt}e^{\alpha t}(\cdot) \qquad (19),$$

so, the problem,

$$L(\theta) = 0 \quad (20)\cdot$$

The inverse operator L^{-1} is therfor,

considerd a tow-fold integral oprator,

as below,

$$L^{-1}(\cdot) = e^{-\alpha t}\int_0^t e^{2\alpha t}\int_0^t e^{-\alpha t}(\theta'' - \alpha^2\theta)(\cdot)dtdt \qquad (21)\cdot$$

Applying L^{-1} of Eq $\cdot$ (21)to the two terms $\theta'' - \alpha^2\theta$ of Eq $\cdot$ (18)

We find

$$L^{-1}(\theta'' - \alpha^2\theta)$$

$$= e^{-\alpha t}\int_0^t e^{2\alpha t}\int_0^t e^{-\alpha t}(\theta'' - \alpha^2\theta)(\cdot)dtdt.$$

$$= e^{-\alpha t}\int_0^t e^{2\alpha t}(e^{-\alpha t}\theta' + \alpha e^{-\alpha t}\theta - \theta'(0) - \alpha\theta(0)dt.$$

$$= \theta(t) - e^{-\alpha t}\theta(0) - \frac{1}{2\alpha}e^{\alpha t}\theta'(0) + \frac{1}{2\alpha}e^{-\alpha t}\theta'(0) - \frac{1}{2}e^{\alpha t}\theta(0) + \frac{1}{2}e^{-\alpha t}\theta(0) \quad \cdot$$

Hence

$$\theta(t) = e^{-i\alpha t}\theta(0) + \frac{1}{2\alpha}e^{\alpha t}\theta'(0) - \frac{1}{2\alpha}e^{-\alpha t}\theta'(0) + \frac{1}{2}e^{\alpha t}\theta(0) - \frac{1}{2}e^{-\alpha t}\theta(0).$$

$$\theta(t) = \cosh\alpha t + \sinh\alpha t \qquad (22)\cdot$$

Example (3-3):

We consider the Schrodinger equation for a simple harmonic oscillator (SHO) The Eq$\cdot$ (4) becomes [3]$\cdot$

$$\varphi''(x) + \beta^2\varphi(x) = 0 \qquad (23)\cdot$$

With intial value $\varphi(0) = 1, \varphi'(0) = \beta,$

we put $\omega = \beta$,substitution of $\beta = \omega$ in Eq $\cdot$ (5) yield the operator

$$L(\cdot) = e^{i\beta x}\frac{d}{dx}e^{-2i\beta x}\frac{d}{dx}e^{i\beta x}(\cdot) \qquad (24),$$

$$L^{-1}(\cdot) = e^{-i\beta x}\int_0^x e^{2i\beta x}\int_0^x e^{-i\beta x}(\cdot)dxdx \qquad (25)\cdot$$

In an operator from, Eq $\cdot$(23)becomes

$$L\varphi = 0 \qquad (26)\cdot$$

Applying L^{-1} on both sides of (23), we find $L^{-1}L\varphi = 0,$

and implies,

$$\varphi(x) = e^{-i\beta x}\varphi(0) + \frac{\varphi'(0)}{2i\beta}e^{i\beta x} - \frac{\varphi'(0)}{2i\beta}e^{-i\beta x} + \frac{\varphi(0)}{2}e^{i\beta x} - \frac{\varphi(0)}{2}e^{-i\beta x},$$

so

$$\varphi(x) = \cos\beta x + \sin\beta x \qquad (27)\cdot$$

4. CONCLUSION

Adomian decomposition method has been known to be a powerful device for solving many functional equations such as algebra equations, ordinary and partial differential equations, integral equations, physical equations and so on. Her we used this method for solving second ordinary differential equations in physical, which called a simple harmonic oscillator. It is demon-stated that this method has the ability of solving equations of both linear and non-linear. Her we used method for solving linear equations, such as a simple pendulum equation, Schrodinger equation. For these equations we derived the exact solution.

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